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M. N. Yakovlev

The authors of [1] and [2] proposed the following method of differentiation with respect to a parameter for solution of systems of nonlinear equations. /146

Let $f(x) = (f_1(x_1,\ldots,x_n),\ f_2(x_1,\ldots,x_n),\ldots,\ f_n(x_1,\ldots,x_n))$ be a vector function of n real variables. In order to find a solution for the system

$$f(x) = 0 \tag{1}$$

of n real equations, we consider a system

$$F(x,\ \lambda) = 0, \tag{2}$$

depending on a real parameter λ, $\lambda \in [0,\ 1]$, in such a manner that $F(x,\ 1) = f(x)$ and a solution to the system $F(x,\ 0) = 0$ exists and can be found easily. Let this solution be x_0; then, in order to find a solution for Eq. (1) we propose to differentiate (2) with respect to λ, assuming that x is a function of λ, and to attempt to solve the Cauchy problem for the differential equation thus obtained with the initial condition $x(0) = x_0$ at the point $\lambda = 1$.

In order to justify this method of solving systems of equations we must prove that the Cauchy problem thus obtained has a solution and, moreover, the solution is unique everywhere in the interval $[0,\ 1]$. Below we will construct such a proof for two methods of introducing the parameter λ.

1. Notation. Below we shall use the following notation and assumptions: G is a domain in the n-dimensional real space R^n; $(x,\ y)$ is the scaler product of elements x and y in R^n, where R^n is Euclidean;

x_0 is a fixed point in G; f(x) is a function in the class $C^1(G)$; $J_f(x)$ (more briefly J(x)) is the Jacobian matrix of the function f; $S(x_0 r)$ is the set of points $x \in R^n$ such that $\| x - x_0 \| \leq r$; $0 \leq r < \infty$; r_0 is the least upper bound of the numbers r such that $S(x_0, r) \subset G$; r* is the least upper bound of the numbers r such that $S(x_0, r) \subset G$ and det $J(x) \neq 0$ (it is clear that $r^* \leq r_0$); D(r) is a function that is continuous and nondecreasing in the interval $0 \leq r \leq r^*$ and is such that $\| J^{-1}(x) \| \leq D(r)$ when $\| x - x_0 \| \leq r < r^*$;

$$\Delta(r) = \int_0^r \frac{1}{D(u)}\, du \quad \text{when} \quad 0 \leq r \leq r^*$$

(if $r^* = \infty$ and the integral $\int_0^{r^*} \frac{1}{D(u)}\, du$ converges, we assume that $\Delta(r^*) = \infty$), $\rho^* = \Delta(r^*)$; $\Delta^{-1}(\rho)$, $0 \leq \rho \leq \rho^*$, is the inverse of $\Delta(r)$.

In the proofs we will give below we will use the following theorem, which was stated by S. M. Lozinskiy (see [3], p. 135, Theorem 1).

Theorem: If $\| f(x_0) \| < \rho^*$, *the equation* $f(x) = 0$ *has a unique solution* x^*, *if* $r^* = \rho^* = \infty$, *and*

$$\| x^* - x_0 \| \leq \Delta^{-1} (\| f(x_0) \|).$$

Below we will use the following

Lemma (uniqueness of solution). If $(J(x)h, h) \geq m(r)(h, h)$, $m(r) > 0$ *for* $\| x - x_0 \| \leq r < r^*$, $h \in R^n$, *the solution to the equation*

$f(x) = 0$ is unique inside the sphere $S(x_0 r^*)$.

2. Theorem 1. Let*

$$F(x, \lambda) \equiv x - x_0 + \lambda\, [f(x) - x + x_0] = 0 \qquad (2.1)$$

and assume that the following conditions are satisfied: /147

(a) $(J(x)h, h) \geq m(r)(h, h)$, $x \in S(x_0, r)$, $m(r) > 0$, is a function that is monotonic and nonincreasing in the interval $0 \leq r \leq r^*$,

(b) $\| f(x_0) \| < \rho^{**}$, where $\rho^{**} = \Delta^*(r^*)$, $\Delta^*(r) = \int_0^r m^*(u)\, du$, $m^*(r) = \min(1, m(r)$

(if $r^* = \infty$ and $\int_0^\infty m^*(u)\, du$ converges, then $\rho^{**} = \infty$). Then the Cauchy problem for the differential equation

$$\frac{dx}{d\lambda} = [(1 - \lambda) E + \lambda f'(x)]^{-1} [x - x_0 - f(x)] \qquad (2.2)$$

with the initial condition $x(0) = x_0$ has a solution, indeed, a unique solution, everywhere in the interval $[0, 1]$.

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*Such a method for introducing a parameter was discussed for special cases earlier in [4] and [5] (see [6] as well). This theorem generalizes (in particular, includes cases in which the equation $f(x) = 0$ has more than one solution) and strengthens (we are relieved of the necessity of determining the behavior of the function $f(x)$ and its derivative $f'(x)$ at infinity; in order to guarantee solvability of the Cauchy problem, the authors of [4] had to carry out such an investigation) the results obtained there.

Proof. We shall first show that Eq. (2.1) uniquely determines the function $x(\lambda)$, which is defined on the interval $0 \le \lambda \le 1$.

For each fixed $\lambda \in [0, 1]$, satisfaction of conditions a) and b) implies satisfaction of the conditions of S. M. Lozinskiy's theorem and the lemma about uniqueness of solutions for the function $F(x, \lambda)$. It follows that for each $\lambda \in [0, 1]$ there exists a u unique solution $x(\lambda)$ for Eq. (2.1), i.e., this also implies that a function satisfying (2.1) is defined on [0, 1] and

$$\|x(\lambda) - x_0\| \quad \int_0^{\|f(x_0)\|} \frac{du}{m^*(u)} < r^*. \tag{2.3}$$

We shall now show that it is differentiable. The conditions of the implicit-function theorem for the function $F(x, \lambda)$ are satisfied at each point $(\lambda, x(\lambda))$. Indeed,

1) $F(x(\lambda), \lambda) = 0$;

2) $[F'_x(x, \lambda)]^{-1}$ exists for all x such that

$$\|x - x_0\| < r^* = \int_0^{\rho^{**}} \frac{du}{m^*(u)} \text{ and } \lambda \in [0, 1]$$

and, therefore, for $(x(\lambda), \lambda)$ (by virtue of (2.3)).

We can use the implicit-function theorem to conclude that in some neighborhood λ there exists, for each $\lambda \in [0, 1]$, some continuous and differentiable function that satisfied Eq. (2.1) and, by virtue of the uniqueness of the solution for this equation, coincides with $x(\lambda)$ for each $\lambda \in [0, 1]$; thus

$$\frac{dx}{d\lambda} = -[F'_x(x, \lambda)]^{-1} F_\lambda(x, \lambda) \tag{2.4}$$

for each point $\lambda \in [0, 1]$.

We shall now prove that the uniqueness of the solution for differential equation (2.2) with the initial value $x(0) = x_0$ for $\lambda \in [0, 1]$. Equation (2.2) is equivalent to Eq. (2.1) for all $\lambda \in [0, 1]$ and functions $x(\lambda)$ whose values belong in the sphere $S(x_0, r^*)$ when $0 \leq \lambda \leq 1$ and are such that $x(0) = x_0$. Expression (2.3) and the uniqueness of the solution to (2.1) imply that the solution of the Cauchy problem for (2.2) with the initial condition $x(0) = x_0$ is unique.

<u>Corollary 1.</u> Assume that the following conditions are satisfied:

a) $(J(x)h, h) \geq m(h, h)$, $m > 0$, where $\|x - x_0\| \leq r \leq r^*$,

b) $\frac{1}{m^*} \| f(x_0)\| < r$, where $m^* = \min \{1, m\}$.

Then the Cauchy problem for (2.2) with the initial condition $x(0) = x_0$ has a solution, indeed, a unique solution, everywhere in the interval $0 \leq \lambda \leq 1$. /148

<u>Corollary 2.</u> If $(J(x)h, h) \geq m(h, h)$, $m > 0$, for all x, $h \in R^n$, the Cauchy problem for (2.2) with the initial condition $x(0) = x_0$ has a solution, indeed, a unique solution, everywhere in the interval $0 \leq \lambda \leq 1$ for any x_0.

3. We shall now consider the other method of introducing the parameter.

<u>Theorem 2. Let</u>

$$F(x, \lambda) = f(x) - (1 - \lambda) f(x_0) = 0; \qquad (3.1)$$

<u>then, if</u>

a) $(j(x)h, h) \geq m(r)(h, h)$, $x \in S(x_0, r)$, $h \in R^n$, $m(r) > 0$ <u>is a function that is nonincreasing in the interval</u> $0 \leq r < r^*$,

b) $\|f(x_0)\| < \rho^*$,

<u>the Cauchy problem for the differential equation</u>

$$\frac{dx}{d\lambda} = - [f'(x)]^{-1} f(x_0) \tag{3.2}$$

<u>with the initial condition</u> $x(0) = x_0$ <u>has a solution, indeed, a unique solution everywhere in the interval</u> $0 \le \lambda \le 1$.

<u>Proof.</u> Again we use S. M. Lozinskiy's theorem and the lemma about the uniqueness of the solution to conclude that there exists a function $x(\lambda)$ that satisfies (3.1) everywhere in $[0, 1]$. In virtue of the implicit-function theorem, whose conditions are satisfied by every point $(\lambda, x(\lambda))$, $\lambda \in [0, 1]$, this function $x(\lambda)$ is continuous and differentiable when $\lambda \in [0, 1]$ and satisfies (3.2). Uniqueness is proved as in Theorem 1.

<u>Corollary 3.</u> If

a) $(j\ (x)\ h,\ h) \ge m\ (h,\ h)$, $m > 0$ for $\|x - x_0\| \le r < r^*$,

b) $\frac{1}{m} \|f(x_0)\| < r$,

then the Cauchy problem for (3.2) with the initial condition $x(0) = x_0$ has a solution, indeed, a unique solution, everywhere in the interval $0 \le \lambda \le 1$.

<u>Theorem 3. If</u> $r^* = \rho^* = \infty$, <u>the Cauchy problem for Eq. (3.2) with the initial condition</u> $x(0) = x_0$ <u>has a solution, indeed, a unique solution, everywhere in the interval</u> $0 \le \lambda \le 1$ <u>for any</u> x_0.

The proof is based on S. M. Lozinskiy's theorem and is similar to the proofs of Theorems 1 and 2.

4. Application of Galerkin's method or the finite-difference method to nonlinear problems leads to the necessity of solving nonlinear systems of a finite number of equations with a finite number of unknowns. In order to solve such systems, we can use the method

of differentiation with respect to a parameter.

We shall now discuss examples.

Assume that on a linear set Ω that is dense in a real Hilbert space H we assign a nonlinear operator $P(x)$ whose domain of values lies in H. Assume that the operator $P(x)$ is weakly Haton differentiable in Ω, and let $(P'(x)h, h) \geq m(h, h)$, $m > 0$; $x, h \in \Omega$. We shall solve the equation $p(x) = 0$ by the Galerkin method, i.e., on taking elements linearly independent $\{\varphi_k\}_{k=1}^n$, $\varphi_k \in \Omega$, as a coordinate system, we will attempt to find an approximate solution in the form

$$u_n = \sum_{k=1}^{n} c_k \varphi_k$$

and determine the c_k from the system of equations $(P(u_n), \varphi_k) = 0$, $k = 1, 2, \ldots, n$. The Jacobian matrix of this system, which is clearly representable in the form $\{(P'(u_n)\varphi_i, \varphi_k)\}_{i,k=1}^n \equiv J$, is invertible for all $c = \{c_k\}_{k=1}^n \in R^n$ and $\|J^{-1}\|_{R^n} \leq 1/m\lambda^{(n)}$, where $\lambda^{(n)}$ is the smallest characteristic value of the Gram matrix of the system $\{\varphi_k\}_{k=1}^n$ and, consequently, when they are solved with the method of differentiation with respect to a parameter, systems (2.2) or (3.2) are solvable everywhere in [0, 1], no matter what the initial value x_0.

Schechter [7] discussed the problem of uniform positive definiteness of the Jacobian matrix of a system of finite difference equations for the case in which the finite-difference method is applied to variational problems. That is, it was exactly he who discussed the variational problem of the minimum $\iint_D F(x, y, p, q)\,dxdy$ under the constraint $u|_\Gamma = b(x, y)$ and derived conditions for uniform positive definiteness of the Jacobian matrix. Note that the result of this paper applies to the case in which the integrand is the more general function $\Phi(x, y, p, q, u) = F(x, y, p, q) + F_1(x, y, u)$, if the following condition is also satisfied: /149

$$\frac{\partial^2}{\partial u^2} F_1 (x, y, u) \geq 0.$$

Comparison of the results of [7] with Corollary 2 or Theorem 3 of the present article makes it possible to assert the solvability of the Cauchy problem for the corresponding systems of differential equations of the form (2.2) or (3.2) everywhere in the interval $0 \leq \lambda \leq 1$ for any initial value x_0.

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